Creation of scalar particles in the presence of a constant electric field in an anisotropic cosmological universe

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Abstract

In the present article we analyze the phenomenon of particle creation in a cosmological anisotropic universe when a constant electric field is present. We compute, via the Bogoliubov transformations, the density number of scalar particles created.

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Quantum field theory in curved space-time is perhaps one of the most interesting and puzzling problems in contemporary theoretical physics. After the publication of the pioneer article by Hawking [1] about pair production in the vicinity of a Schwarzschild black hole, a great body of papers have been published, mainly trying to understand the mechanism that gives origin to the thermal particle distribution and its relation to thermodynamics. It is noteworthy that Hawking's result was preceded by a series of articles where the question was to discuss particle production in cosmological universes [2,3]. Almost all of the articles published in this area deal with isotropic and homogeneous gravitational backgrounds, mainly in deSitter and Robertson Walker models, and only a few try to discuss quantum processes in anisotropic Universes.

The study of quantum effects in gravitational backgrounds with initial singularities presents an additional difficulty. The techniques commonly applied in order to define particle states are based on the existence of a timelike Killing vector or an asymptotically static metric [4]. A different approach is needed to circumvent the problem related to the initial singularity. In this direction, the Feynman path-integral method has been applied to the quantization of a scalar field moving in the The Chitre-Hartle Universe [5,6]. This model has a curvature singularity at $t = 0$, and it is perhaps the best known example where a time singularity appears and consequently any adiabatic prescription in order to define particle states fails. A spin 1/2 extension has been considered by Sahni. [7]

A different approach to the problem of classifying single particle states on curved spaces, is based on the idea of diagonalizing the Hamiltonian. This technique permits one to compute the mean number of particles produced by a singular cosmological model, and in particular by the Chitre-Hartle Universe [5].

An interesting scenario for discussing particle creation processes is the anisotropic universe associated with the metric

$$ds^2 = -dt^2 + t^2(dx^2 + dy^2) + dz^2. \tag{1}$$

The line element (1) presents a space-like singularity at $t = 0$. The scalar curvature is $R = 2/t^2$, and consequently, the adiabatic approach [4]cannot be applied in order to define particle states. With the help of the Hamiltonian diagonalization method [8–10], Bukhbinder [11] has been able to compute the rate of scalar particles produced in the space with the metric (1), obtaining as result a Bose-Einstein distribution. More recently [12], a quasiclassical approach has been applied to compute the rate of scalar as well as Dirac particles in the metric (1).

The introduction of an external electric field permits one to consider an additional source of quantum processes. The density of particles created by an intense electric field was first calculated by Schwinger [13], different authors [8,14] have discussed this problem. Pair creation of scalar particles by a constant electric field in a 2+1 de Sitter cosmological universe has been analyzed by Garriga [15]. Quantum effects associated with scalar and spinor particles in a quasi-Euclidean cosmological model with a constant electric field are discussed by Bukhbinder and Odintsov [16]. It is the purpose of the present article to compute, via the quasiclassical approach [17–19], the density of scalar particles created in the background field (1) when a constant electric field is present. The idea behind the method is the following: First, we solve the relativistic Hamilton-Jacobi equation and, looking at its solutions, we identify positive and negative frequency modes. Second, we solve the Klein Gordon-equation and, after comparing with the results obtained for the quasiclassical limit, we identify the positive and negative frequency states. This technique has already been successfully applied in different scenarios [17–19].

The relativistic Hamilton-Jacobi equation can be written as

$$g^{\alpha\beta}(\frac{\partial S}{\partial x^{\alpha}} - eA_{\alpha})(\frac{\partial S}{\partial x^{\beta}} - eA_{\beta}) + m^2 = 0, \tag{2}$$

where here and elsewhere we adopt the convention $c = 1$ and $\hbar = 1$.

The vector potential A_α

$$A_\alpha = (0, 0, 0 - Et), \tag{3}$$

corresponds to a constant electric field $E\hat{k}$. The corresponding invariants $F^{\mu\upsilon}F_{\mu\nu} = -2E^2$ and $F^{\mu\upsilon *}F_{\mu\nu} = 0$ indicate that there is no magnetic field. Since the metric $g_{\alpha\beta}$ associated with the line element (1) only depends on t, the function S can be separated as

$$S = F(t) + k_x x + k_y y + k_z z. \tag{4}$$

Substituting (4) into (2) we obtain

$$\dot{F}^2 = \frac{k_x^2 + k_y^2}{t^2} + (k_z + eEt)^2 + m^2 \tag{5}$$

the solution of Eq. (5) presents the following asymptotic behavior:

$$\lim_{t\to\infty} F = \pm\frac{1}{2}t\sqrt{e^2E^2t^2 - m^2} \mp \frac{m^2}{2eE}\log(eEt + \sqrt{e^2E^2t^2 - m^2}), \tag{6}$$

$$\Phi = e^{iS} \to Ce^{\pm\frac{i}{2}eEt^2}(eEt)^{\mp\frac{im^2}{2eE}} \tag{7}$$

as $t \to \infty$, and

$$\lim_{t\to 0} F = \pm\sqrt{(k_x^2 + k_y^2)}\log t,\ \Phi = e^{iS} \to Ct^{\pm i\sqrt{k_x^2+k_y^2}}, \tag{8}$$

as $t \to 0$, that is, in the initial singularity. Notice that the time dependence of the relativistic wave function is obtained via the exponential operation $\Phi \to \exp(iS)$. Here it is worth mentioning that the behavior of positive and negative frequency states is selected depending on the sign of the operator $i\partial_t$. Positive frequency modes will have positive eigenvalues and for negative frequency states we will have the opposite. Then in Eq. (6) and (8,7) upper signs are associated with negative frequency values and the lower signs correspond to positive frequency states. After making this identification we can analyze the solutions of the Klein-Gordon equation in the background field (1).

The covariant generalization of the Klein-Gordon equation takes the form

$$g^{\alpha\beta}(\nabla_\alpha - ieA_\alpha)(\nabla_\beta - ieA_\beta)\Phi - (m^2 + \xi R)\Phi = 0, \tag{9}$$

where ∇_α is the covariant derivative, R is the scalar curvature, and ξ is a dimensionless coupling constant which takes the value $\xi = 1/6$ in the conformal case, and $\xi = 0$ when a minimal coupling is considered. Substituting (1) into (9) we obtain

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial z^2} - 2eEt\frac{\partial\Phi}{\partial t} - e^2E^2t^2\Phi - \frac{1}{t^2}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + (m^2 + \frac{2\xi}{t^2})\Phi = 0. \tag{10}$$

Since eq. (10) commutes with the linear momentum $\vec{p} = (-i\partial_x, -i\partial_y - i\partial_z)$, we have that the substitution

$$\Phi = t^{-1}\Delta(t)e^{i(k_x x + k_y y + k_z z)}, \tag{11}$$

reduces eq (10) to the ordinary second order differential equation.

$$\frac{d^2\Delta}{dt^2} + \left(\frac{1}{t^2}\left(k_x^2 + k_y^2 + 2\xi\right) + t^2e^2E^2 + 2tk_zE + k_z^2 + m^2\right)\Delta = 0 \tag{12}$$

whose solution, for $k_z = 0$, can be expressed in terms of Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ [20,21]

$$\Delta = z^{-1/4}(C_1M_{k,\mu}(z) + C_2W_{k,\mu}(z)), \tag{13}$$

where k, μ and z are given by the expressions

$$z = ieEt^2, \quad k = -i\frac{m^2}{4eE}, \quad \mu = \frac{i}{4}\sqrt{4(2\xi + k_y^2 + k_x^2) - 1}. \tag{14}$$

Looking at the asymptotic behavior of $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ as $|z| \to \infty$

$$W_{k,\mu}(z) \sim e^{-z/2}z^k, \tag{15}$$

and as $z \to 0$

$$M_{k,\mu}(z) \sim e^{-z/2}z^{1/2+\mu}, \tag{16}$$

we obtain that the solution (13) having the asymptotic behavior given by (6) and (7) is

$$\Delta_\infty^+ = C_\infty^+ z^{-1/4}W_{k,\mu}(z), \quad \Delta_\infty^- = C_\infty^- z^{-1/4}W_{-k,\mu}(-z), \tag{17}$$

where C_∞^+ and C_∞^- are normalization constants

Analogously, we have that in the vicinity of the singularity, looking at the quasiclassical solutions at $t = 0$ (8) the corresponding negative "$-$" and positive "+" frequency solutions take the form

$$\Delta_0^- = C_0^- z^{-1/4}M_{k,\mu}(z), \quad \Delta_0^+ = C_0^+ z^{-1/4}M_{k,-\mu}(z), \tag{18}$$

where C_0^- and C_0^+ are normalization constants, and the function $M_{k,\mu}(z)$ can be expressed in terms of the Kummer hypergeometric function $M(a, b, z)$ as follows:

$$M_{k,\mu}(z) = e^{-z/2}z^{1/2+\mu}M(\frac{1}{2} + \mu - k, 1 + 2\mu, z). \tag{19}$$

The Whittaker function $M_{k,\mu}(z)$ can be expressed in terms of $W_{k,\mu}(z)$ as [23]

$$M_{k,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu-k+\frac{1}{2})}e^{-i\pi k}W_{-k,\mu}(-z) + \frac{\Gamma(2\mu+1)}{\Gamma(\mu+k+\frac{1}{2})}e^{-i\pi(k-\mu-\frac{1}{2})}W_{k,\mu}(z). \tag{20}$$

Using the above expression (20) we have that the negative frequency solution Δ_0^- can be written in terms of Δ_∞^+ and (Δ_∞^-) as follows

$$\Delta_0^- = \frac{\Gamma(2\mu+1)}{\Gamma(\mu-k+\frac{1}{2})}e^{-i\pi k}\Delta_\infty^- + \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+\mu+k)}(-1)^{-1/4}e^{-i\pi\mu(k-\mu-\frac{1}{2})}(\Delta_\infty^-)^* \tag{21}$$

where we have made use of the property $W_{-k,\mu}(-z) = (W_{k,\mu}(z))^*$

Since we have been able to obtain single particle states for in the vicinity of $t = 0$ as well as in the asymptote $t \to \infty$, we can compute the density of particles created by the gravitational field. With the help of the Bogoliubov coefficients [4,8]. From (21) and the fact that $\Delta_0^- = \alpha\Delta_\infty^- + \beta(\Delta_\infty^-)^*$ we obtain

$$\frac{|\beta|^2}{|\alpha|^2} = e^{2i\pi\mu}\frac{\left|\Gamma(\frac{1}{2}+\mu-k)\right|^2}{\left|\Gamma(\frac{1}{2}+\mu+k)\right|^2}. \tag{22}$$

Substituting into (22) the values for μ and k we obtain

$$\frac{|\beta|^2}{|\alpha|^2} = \frac{\cosh(\frac{\pi}{4}\sqrt{4(2\xi+k_y^2+k_x^2)-1} - \frac{\pi m^2}{4eE})}{\cosh(\frac{\pi}{4}\sqrt{4(2\xi+k_y^2+k_x^2)-1} + \frac{\pi m^2}{4eE})}e^{-\frac{\pi}{2}\sqrt{4(2\xi+k_y^2+k_x^2)-1}} \tag{23}$$

where we have used the relation [20]

$$\left|\Gamma(\frac{1}{2}+iy)\right|^2 = \frac{\pi}{\cosh \pi y}. \tag{24}$$

The computation of the density of particles created is straightforward from (23) and the normalization condition [22] of the wave function

$$|\alpha|^2 - |\beta|^2 = 1, \tag{25}$$

then

$$n = |\beta|^2 = \left[\left(\frac{|\beta|^2}{|\alpha|^2}\right)^{-1} - 1\right]^{-1}.$$

It should be noticed that, thanks to the normalization condition, we did not have to compute the normalization constants C appearing in the definition of the single mode solutions (17,18). Let us analyze the asymptotic behavior of (23) when the electric field vanishes. Taking into account that $cosh(z) \sim e^{|z|}/2$ as $z \to \infty$, we readily obtain

$$n \sim \frac{|\beta|^2}{|\alpha|^2} = \exp(-\pi\sqrt{4(2\xi+k_y^2+k_x^2)-1}), \tag{26}$$

which is the result obtained in [12]. Expression (26) corresponds to a two dimensional Bose-Einstein thermal distribution with an effective mass which differs from the value of m appearing in Eq. (9). In the case of strong electric fields the density number of scalar particles created takes the form

$$n \approx \frac{|\beta|^2}{|\alpha|^2} = \exp(-\frac{\pi}{2}\tanh(\frac{\pi}{4}\sqrt{4(2\xi + k_y^2 + k_x^2) - 1})\frac{m^2}{eE} - \frac{\pi}{2}\sqrt{4(2\xi + k_y^2 + k_x^2) - 1}), \quad (27)$$

showing that the density of particles created by the cosmological background and the electric field (27) is a Bose-Einsten distribution with a chemical potential proportional to $\frac{m^2}{eE}$. Integrating the particle density n (27) on momentum we obtain the total number of particles created per unit volume.

$$N = \frac{1}{V}\int n dk_x dk_y dk_z = \frac{1}{t^2(2\pi)^2}\int n k_\perp dk_\perp dk_z. \quad (28)$$

In order to carry out the integration we have to notice that n does not depend on k_z and consequently integration on k_z is equivalent to the substitution [8,14] $\int dk_z \rightarrow eET$ where T is the time of interaction of the external field. In the strong field limit we can approximate the density number n (27) by the expression

$$n \approx \exp(-\frac{\pi}{2}\tanh(\frac{\pi}{4}\sqrt{8\xi - 1})\frac{m^2}{eE} - \frac{\pi}{2}\sqrt{4(2\xi + k_y^2 + k_x^2) - 1}). \quad (29)$$

Substituting (29) into (28) , we obtain that the total number N of particles per unit volume takes the form

$$N \approx \frac{eE}{8\pi^4 T}(b+2)\exp(-b/2)\exp(-\frac{m^2\pi}{2eE}\tanh(\frac{b}{4})), \quad (30)$$

where $b = \pi\sqrt{8\xi - 1}.$ Result (30) resembles the number of particles created by a constant electric field in a Minkowski space [8,14]. It is worth mentioning that the number N of particles per unit volume is inversely proportional to T^{-1} and vanishes as $T \rightarrow \infty$. The volume expansion of the anisotropic universe (1) is faster than the particle creation process, therefore N becomes negligible for large values of T.

The results (23) , (27), and (30) show that the anisotropic cosmological background (1), as well the constant electric field, contribute to the creation of scalar particles. The quasiclassical method gives a recipe for obtaining the positive and negative frequency modes even when spacetime is not static and an external source is present. The presence of the anisotropy with a constant electric field gives place to a particle distribution that is thermal only in the asymptotic field regime. The method and results presented in this paper could be of help to discuss quantum effects in more realistic anisotropic cosmological scenarios.

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REFERENCES

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975)
[2] L. Parker Phys. Rev . Lett. **21**, 562 (1968)
[3] Ya. Zeldovich and A. A. Starobinskii Sov. Phys. JETP **34**, 1159 (1971)
[4] N. D. Birrel and P. C. W. Davies *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press, 1982)
[5] D. M. Chitre and J. B. Hartle Phys. Rev D. **16**, 251 (1977)
[6] M. V. Fischetti, J. B. Hartle, and B. L. Hu Phys. Rev. D **20**, 1757 (1979)
[7] V. Sahni Class. Quantum Grav. **1**, 579 (1984)
[8] A. A. Grib, S. G. Mamaev and V. M. Mostepanenko *Quantum Vacuum effects in strong fields* (Moscow: Energoatomizdat, 1988)
[9] I. L. Bukhbinder, and D. M. Gitman Izv. Vuzov Fizika **3**, 90 (1979)
[10] I. L. Bukhbinder, and D. M. Gitman Izv. Vuzov Fizika **4**, 55 (1979)
[11] I. L. Bukhbinder Izv. Vuzov. Fizika **7**, 3 (1980)
[12] V. M. Villalba Int. J. Theor. Phys. **36**, 1321 (1997)
[13] J. Schwinger, Phys. Rev. **82**, 664 (1951)
[14] A. I. Nikishov, *Trudy FIAN No. 111* (Nauka, Moscow, 1979).
[15] J. Garriga, Phys. Rev. D, 49 **6343** (1994)
[16] I. L. Bukhbinder and S. Odintsov Izv. Vuzov. Fizika **5**, 12 (1982)
[17] I. Costa J. Math. Phys. **30**, 888 (1989)
[18] U. Percoco and V. M. Villalba Class. Quantum Grav. **9**, 307 (1992)
[19] V. M. Villalba, Phys. Rev D. **52**, 3742 (1995)
[20] N. N. Lebedev *Special Functions and their applications* (New York: Dover, 1972)
[21] M. Abramowitz and I. Stegun *Handbook of Mathematical Functions* (Dover, New York, 1974)
[22] T. Mishima and A. Nakayama Phys. Rev D. **37** 348 (1988)
[23] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York 1980)